

# Effects of a nonlinear bath at low temperatures

Hanno Gassmann and C. Bruder

*Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland*

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We use the numerical flow-equation renormalization method to study a nonlinear bath at low temperatures. The model of our nonlinear bath consists of a single two-level system coupled to a linear oscillator bath. The effects of this nonlinear bath are analyzed by coupling it to a spin, whose relaxational dynamics under the action of the bath is studied by calculating spin-spin correlation functions. As a first result, we derive flow equations for a general four-level system coupled to an oscillator bath, valid at low temperatures. We then treat the two-level system coupled to our nonlinear bath as a special case of the dissipative four-level system. We compare the effects of the nonlinear bath with those obtained using an effective linear bath, and study the differences between the two cases at low temperatures.

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## I. INTRODUCTION

The linear bath of harmonic oscillators is the most common model system to describe a quantum environment or bath, leading to dissipation and decoherence<sup>1,2,3</sup>. In this paper we examine an environment which is the simplest possible quantum-mechanical non-oscillator bath<sup>4</sup>: a single two-level system subject to an oscillator bath. The effects of this nonlinear bath are analyzed by coupling it to a spin, whose relaxational dynamics under the action of the bath is studied by calculating spin-spin correlation functions.

Nonlinear baths have received a great deal of interest in the field of quantum computation, e.g., in the context of superconducting devices<sup>5,6</sup> and for spin-based proposals<sup>7,8,9</sup>.

There are two ways to define the border between system and bath in our total system. The first is to consider one two-level system to be the system and the other two-level system belonging to the bath. The bath consists then of a two-level system and the oscillators together and therefore it is a non-oscillator or nonlinear bath. Studying the problem this way is relevant, since, in many physical situations where the precise nature of the bath decohering a given system is unknown, it is modeled as a linear bath, with some given correlation function. It is, therefore, desirable to understand in more detail the kind and magnitude of possible errors introduced by such an approximation, in cases where the coupling cannot be assumed to be weak.

The other way to define the system and bath is as follows: taken together, the two-level systems form a four-level system that is coupled to the linear oscillator bath. Such systems of two two-level systems coupled to linear oscillator baths, representing a formidable problem in itself, have been studied in-depth using different methods, such as functional-integral approaches<sup>10</sup>, master equations<sup>11,12</sup>, and numerical calculations within the quasi-adiabatic propagator path integral method<sup>13</sup>. In this work we use a different approach: the numerical flow-equation renormalization method.

The flow-equation method itself was introduced by Wegner<sup>14</sup> and by Glasek and Wilson<sup>15</sup>. Using the flow-equation method one works in a Hamiltonian framework. The whole procedure is non-perturbative, as it relies on a unitary transformation and has an energy-scale separation built in. The extension of the flow-equation formalism to dissipative systems was developed in Refs. 16, 17, and 18. The effective Hamiltonian simplifies due to the fact that the coupling between bath and system disappears and equilibrium functions can be calculated easily. The method is non-perturbative, e.g., neither a Born nor Markoffian approximation is applied, which is what makes it attractive and new. Some drawbacks are that the flow equations for the observables are not closed, and a special ansatz is needed. Here, we choose a linear ansatz, which works only for low temperatures smaller than the typical low-energy scale of the system Hamiltonian. What is new in our work is that we apply the method to a general dissipative four-state system. The flow equations are designed to yield correlation functions for non-ohmic baths as well, which, as we will see below, makes it possible to treat the linear bath approximation to our nonlinear problem on the same footing. In Refs. 19, 20, 21, 22, and 23, subohmic, superohmic, and a peaked structured environment, were studied with flow equations. Similarly, the linear-bath approximation leads to a new peaked structured bath spectral density, to our knowledge, never studied before. The numerical integration of the flow equations allows us to study the nonlinear bath model by analyzing the spin-spin correlation functions. The numerical results are obtained at zero temperature. These results are then compared with those obtained from a linear bath with a non-ohmic bath spectral density. The bath spectral density is chosen in such a way that the linear bath leads to the same results as the original nonlinear bath in the limit of small enough system-bath coupling strength. The bath and system correlation functions are both calculated using flow equations. By carrying out the comparison, we are able to study the effects of the nonlinear bath at zero temperature.

The remainder of this work is organized as follows: First we describe the model in Sec. II. Then, we write down the flow equations for the general dissipative four-level system. The flow equations for the nonlinear-bath system (Sec. III) and its linear approximation (Sec. IV) are both special cases of these general flow equations. Finally, we discuss the numerical details in Sec. V and compare the results of the two approaches in Sec. VI.

## II. THE NONLINEAR BATH MODEL

As a model for a nonlinear bath we study a two-level system  $S$  coupled to a bath, which consists of another two-level system  $B$  coupling to a linear oscillator bath  $F$ :

$$H = \epsilon_S \sigma_z^S + \Delta_S \sigma_x^S + g \sigma_z^S \sigma_z^B + \Delta \sigma_x^B + \sigma_z^B \sum_k \lambda_k (b_k + b_k^\dagger) + \sum_k \omega_k : b_k^\dagger b_k : . \quad (2.1)$$

Here the parameters  $\epsilon_S$  and  $\Delta_S$  serve to define any desired two-level system  $S$ . This system is coupled to  $B$  via  $\sigma_z^B$ , with the coupling strength between  $S$  and  $B$  being given by  $g$ . The coupling constants  $\lambda_k$  are connected to the bath spectral density in the following way:

$$J(\omega) \equiv \pi \sum_k \lambda_k^2 \delta(\omega - \omega_k) = \alpha \omega f(\omega/\omega_c), \quad (2.2)$$

with the bath cutoff  $\omega_c$ , the system-bath coupling strength  $\alpha$  and the cutoff function  $f(x)$ . The oscillations of  $\sigma_z^B(t)$  at the frequency  $2\Delta$  are noisy, due to the action of the harmonic oscillator bath  $F$ . The dissipative dynamics of  $S$  can be characterized in terms of several different quantities. Here we will analyze the decay of the equilibrium correlator  $\langle \sigma_z^S(t) \sigma_z^S(0) \rangle$  by calculating its Fourier transform:

$$K_{zz}^S(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \sigma_z^S(t) \sigma_z^S(0) \rangle . \quad (2.3)$$

$K_{zz}^S(\omega)$  is real-valued, for zero temperature it vanishes on the negative real axis and the integral over all frequencies gives 1.

## III. FLOW EQUATIONS FOR THE DISSIPATIVE FOUR-STATE SYSTEM

We will write down the flow equations for the most general four-level system coupling to a linear oscillator bath. Therefore many other systems can be analyzed as special cases of this general framework. The Hamiltonian given in Eq. (2.1) is a special case of:

$$H = \sum_{\alpha\beta} \Delta^{\alpha\beta} \Sigma_{\alpha\beta} + \sum_k \sum_{\alpha\beta} \lambda_k^{\alpha\beta} \Sigma_{\alpha\beta} (b_k + b_k^\dagger) + i \sum_k \sum_{\alpha\beta} \kappa_k^{\alpha\beta} \Sigma_{\alpha\beta} (b_k - b_k^\dagger) + \sum_k \omega_k : b_k^\dagger b_k : . \quad (3.1)$$

The  $\Sigma_{\alpha\beta}$  are the tensor products of the Pauli matrices  $\Sigma_{\alpha\beta} = \sigma_\alpha \otimes \sigma_\beta$  and the Greek indices are always summed from zero to three.  $\sigma_0$  is the unit matrix. By using the flow-equation technique we approximately diagonalize the model Hamiltonian given in Eq. (3.1) by means of a unitary transformation:

$$H(l) = U(l) H U(l)^\dagger . \quad (3.2)$$

Here,  $l$  is the flow parameter which is roughly equivalent to the square of the inverse energy scale which is being decoupled. The unitary transformation can be written in a differential form

$$\frac{dH(l)}{dl} = [\eta(l), H(l)] , \quad (3.3)$$

where

$$\eta(l) = \frac{dU(l)}{dl} U^{-1}(l) . \quad (3.4)$$

Using the anti-Hermitian generator  $\eta$  in the canonical form  $\eta = [H_0, H]$  leads to the fixed point  $H(l \rightarrow \infty) = H_0$ . We have investigated two different generators  $\tilde{\eta}$  and  $\eta$ , leading to two different fixed points, viz.,

$$\tilde{H}_0 = \sum_k \omega_k : b_k^\dagger b_k : \quad (3.5)$$

the oscillator bath, and

$$H_0 = \sum_{\alpha\beta} \Delta^{\alpha\beta} \Sigma_{\alpha\beta} + \sum_k \omega_k : b_k^\dagger b_k : \quad (3.6)$$

the total Hamiltonian without interaction terms. Note that we only have to derive flow equations with the generator  $\eta$ . The flow equations using  $\tilde{\eta}$  are contained in the latter, due to the similar structure of the generators.

The commutator  $[\eta, H]$  contains coupling terms which are bilinear in the bosonic operators. We neglect these terms by truncating the Hamiltonian after linear bosonic terms. The bilinear terms are included by modifying the generator with an additional bilinear term as follows<sup>17,21</sup>:

$$\eta = [H_0, H] + \sum_{kq} \eta_{kq} : (b_k + b_k^\dagger)(b_q - b_q^\dagger) : . \quad (3.7)$$

The parameters  $\eta_{kq}$  are chosen in such a way that the bilinear terms are not generated. Because this cannot be done exactly we neglect terms which have the normal-ordered form: system operator times a bilinear bosonic operator. Here the normal order is defined with respect to the non-interacting Hamiltonian.

Comparing the general Hamiltonian given in Eq. (3.1) with Eq. (2.1) specifies the initial conditions for the flow equations listed in Appendices A and B. The flow equations are nonlinear ordinary coupled differential equations and they have the same structure as those obtained in Ref. 21 due to the similar truncation procedure. The

renormalization of the bath modes  $\omega_k$  vanishes in the thermodynamic limit and can be neglected<sup>16</sup>.

In order to calculate correlation functions the observables have to be subjected to the same sequence of infinitesimal transformations as the Hamiltonian. The observable flow cannot be closed and therefore a linear ansatz has to be chosen which is only valid for temperatures smaller than the typical low-energy scale of the four-level system coupling to the linear oscillator bath. The most general linear ansatz which is possible and used in the following is given by:

$$O = \sum_{\alpha\beta} h^{\alpha\beta} \Sigma_{\alpha\beta} + \sum_k \sum_{\alpha\beta} \mu_k^{\alpha\beta} \Sigma_{\alpha\beta} (b_k + b_k^\dagger) + i \sum_k \sum_{\alpha\beta} \nu_k^{\alpha\beta} \Sigma_{\alpha\beta} (b_k - b_k^\dagger). \quad (3.8)$$

The flow equations for the observables

$$\frac{dO(l)}{dl} = [\eta(l), O(l)] \quad (3.9)$$

are closed according to the same normal ordering scheme as above and are listed in the Appendix C. The correlation functions for  $l \rightarrow \infty$  are defined by:

$$\langle O(t)O \rangle = \frac{\text{tr}\{e^{-\beta H_0} e^{itH_0} O e^{-itH_0}\}}{\text{tr}\{e^{-\beta H_0}\}}. \quad (3.10)$$

With the non-interacting fixed-point Hamiltonian  $H_0$  such correlation functions can be calculated easily.

Below we present the comparatively short result for the correlation functions for the fixed point  $\tilde{H}_0$ . They are found to be:

$$\langle O(t)O \rangle = h^{00}h^{00} + h^{0i}h^{0i} + h^{i0}h^{i0} + h^{ij}h^{ij} + \sum_k \left[ (n_k + 1)e^{-i\omega_k t} c_k + n_k e^{i\omega_k t} c_k \right], \quad (3.11)$$

where

$$\begin{aligned} c_k = & \mu_k^{00}\mu_k^{00} + \mu_k^{0i}\mu_k^{0i} + \mu_k^{i0}\mu_k^{i0} + \mu_k^{ij}\mu_k^{ij} \\ & + \nu_k^{00}\mu_k^{00} + \nu_k^{0i}\mu_k^{0i} + \nu_k^{i0}\mu_k^{i0} + \nu_k^{ij}\mu_k^{ij} \\ & - \mu_k^{00}\nu_k^{00} - \mu_k^{0i}\nu_k^{0i} - \mu_k^{i0}\nu_k^{i0} - \mu_k^{ij}\nu_k^{ij} \\ & - \nu_k^{00}\nu_k^{00} - \nu_k^{0i}\nu_k^{0i} - \nu_k^{i0}\nu_k^{i0} - \nu_k^{ij}\nu_k^{ij}. \end{aligned} \quad (3.12)$$

The Fourier transform of the correlation function is given by

$$\langle OO \rangle_\omega = (h^{00}h^{00} + h^{0i}h^{0i} + h^{i0}h^{i0} + h^{ij}h^{ij})\delta(\omega) + \sum_k \left[ (n_k + 1)\delta(\omega - \omega_k)c_k + n_k\delta(\omega + \omega_k)c_k \right], \quad (3.13)$$

where the Fourier transform is defined by

$$\langle OO \rangle_\omega \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle O(t)O \rangle. \quad (3.14)$$

With these formulas different correlation functions can be calculated. The above formulas can also be used to obtain correlation functions for different operators, i.e.,  $\langle O^{(1)}(t)O^{(2)} \rangle$ . The only difference in the result for  $c_k$  is that now a sum of products of the form  $\mu_k^{(1)0i}\mu_k^{(2)0i}$  occurs and therefore the number of flow equations is increased. Below we put  $O = \sigma_z^S$ , i.e., analyze the decay of the equilibrium correlator  $\langle \sigma_z^S(t)\sigma_z^S(0) \rangle$ . The correlation function is determined by the initial conditions of the flow equations. The limit of zero temperature  $T = 0$  can be obtained by putting  $n_k = 0$  in the flow equations.

#### IV. THE LINEAR-BATH APPROXIMATION

In this section we study an approximation to the non-linear bath. We do this by replacing the composite bath  $B$  and  $F$  by a linear oscillator bath acting on the system  $S$ . What we obtain is a spin-boson system with a non-ohmic peaked bath spectral density

$$H = \Delta_S \sigma_x^S + \epsilon_S \sigma_z^S + \sigma_z^S \sum_k \tilde{\lambda}_k (b_k + b_k^\dagger) + \sum_k \omega_k : b_k^\dagger b_k :, \quad (4.1)$$

where the bath spectral density is given by

$$\tilde{J}(\omega) \equiv \pi \langle BB \rangle_\omega \equiv \pi \sum_k \tilde{\lambda}_k^2 \delta(\omega - \omega_k). \quad (4.2)$$

The linear-bath approximation should become exact in the weak-coupling limit  $g \ll 1$  according to Ref. 24. For higher coupling  $g$  we expect deviations to the nonlinear bath even in the thermodynamic limit  $N \rightarrow \infty$  due to, e.g., the non-separable structure<sup>25</sup> of our nonlinear bath model.

We start our discussion by calculating the bath correlation function. The Fourier transform of the correlator of  $B \equiv g\sigma_z^B$  defines the ‘‘bath spectrum’’  $\langle BB \rangle_\omega$ . For zero temperature,  $\langle BB \rangle_\omega$  vanishes for negative frequencies. The Hamiltonian of the bath alone is given by

$$H_{BF} = \Delta \sigma_x^B + \sigma_z^B \sum_k \lambda_k (b_k + b_k^\dagger) + \sum_k \omega_k : b_k^\dagger b_k :. \quad (4.3)$$

This is the initial Hamiltonian for the flow equations. For the purpose of calculating the bath correlation function, one has to solve the symmetric spin-boson problem. This cannot be done in a closed analytical way<sup>3,26,27</sup>. An alternative method involves flow equations, which we apply again. We use the flow equation system used in Ref. 21 with a generator of the form  $\tilde{\eta}$ . The truncation scheme includes (as before) all coupling terms which are linear in the bosonic modes. The flow equation method is not restricted to any particular bath type, i.e., the bath spectral density of the linear bath can be chosen freely.

The results for the bath correlation function  $\langle BB \rangle_\omega$  are shown in Fig. 1.  $\langle BB \rangle_\omega$  consists of one peak at the frequency  $\approx 2\Delta$ , for small coupling  $\alpha$ . For larger coupling it is shifted towards lower frequencies. Because the

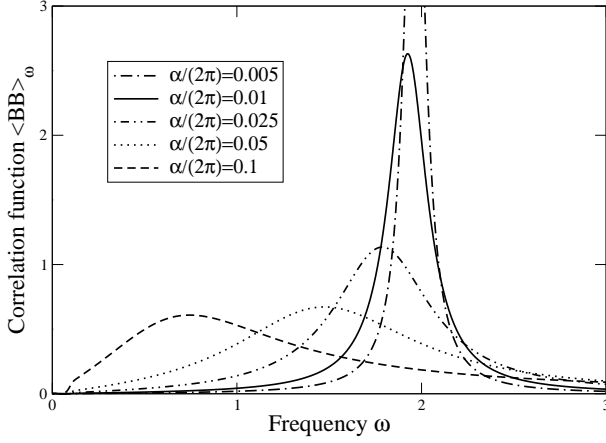


FIG. 1: The  $\langle BB \rangle_\omega$  correlation function for different values of  $\alpha$ . The parameters are:  $g = 1$  and  $\Delta = 1$ . The height of the  $\alpha = 0.005$  peak is  $\sim 5.2$ .

bath  $F$  is ohmic, the position of the maximum of  $\langle BB \rangle_\omega$  is given approximately by  $2\Delta(2\Delta/\omega_c)^{2\alpha/(\pi-2\alpha)}$ , which is a result obtained within an adiabatic renormalization scheme<sup>3</sup>. In our results, the relation is best fulfilled for  $\alpha/(2\pi) \lesssim 0.025$ . The low-frequency behavior predicted by the exact zero-temperature Shiba relation can only be found by solving the asymptotic flow equations<sup>21</sup>, which is not done in this work (note the deviation from linearity for small  $\omega \ll 1$  in Fig. 1, e.g., graph for  $\alpha/(2\pi) = 0.1$ ). Thus, the bath spectral density shows a resonance at a characteristic frequency and behaves ohmic at small frequencies. This is why we expect the results to be comparable to the ones obtained in Ref. 23,28. But we stress that here, in contrast, the linear bath is modified with a second two-state system, compared to an additional oscillator.

To obtain the final result for the linear-bath approximation we use the  $\langle BB \rangle_\omega$  correlation function as bath spectrum  $\tilde{J}(\omega) = \pi \langle BB \rangle_\omega$  and solve the flow equations for the spin-boson system with this non-ohmic spectrum. The results are discussed in Section VI.

To close this section, we make a short note regarding higher-order correlation functions with respect to the flow equations used here. Higher-order correlation functions are important to understand the behavior of a nonlinear bath. For small enough system-bath coupling strength, the nonlinear bath is identical to a linear one<sup>24</sup>. In this limit, the two-time bath correlation function plays a crucial role. For higher coupling strength, higher-order correlation functions become relevant. The linear bath has the special feature that higher-order correlation functions factorize into two-time correlation functions, since due to the Gaussian nature of the linear bath, the cumulant expansion breaks off at second order. The nonlinear case has generally no such feature, but we will see that the correlation functions obtained with the flow equations factorize. The reason for this is the choice of a

linear ansatz for the observable flow:

$$\sigma_z(l) = h^{03}(l)\sigma_z + \mu_k^{01}(l)\sigma_x(b_k + b_k^\dagger). \quad (4.4)$$

In the limit  $l \rightarrow \infty$  we find for the symmetric spin-boson system

$$\sigma_z(l \rightarrow \infty) = \mu_k^{01}(l \rightarrow \infty)\sigma_x(b_k + b_k^\dagger), \quad (4.5)$$

and  $H(l \rightarrow \infty) = \tilde{H}_0 = \sum_k \omega_k b_k^\dagger b_k$ . For  $l \rightarrow \infty$  the observables behave like linear oscillator variables and all higher-order correlation functions factorize.

## V. DETAILS OF THE NUMERICAL CALCULATION

The flow equations are coupled ordinary differential equations. The integration method which turns out to be the most appropriate one is the fourth-order Runge-Kutta method with variable step size<sup>29</sup>. The reason for this is that there are regions where a certain subset of the differential equations does not change much anymore and larger integration steps can be used, see discussion below.

We have used a linear energy spacing  $\delta\omega$  for the bath modes  $\omega_k$  and a sharp cutoff function  $f(x) = \Theta(1 - x)$ . The results do not depend on these choices. The bath modes  $\omega_k$  are then given by  $\omega_k = (k - 1/2)\delta\omega$  and the coupling constants  $\lambda_k$  are found to be  $\lambda_k = (\sqrt{\alpha/\pi}\sqrt{k - 1/2})\delta\omega$ , where  $\delta\omega = \omega_c/N$  and  $N$  is the number of bath modes used for the numerics. For the four-level system the number of differential equations is  $64N + 32$  in the most general case,  $5N + 7$  for the nonlinear bath with  $\epsilon_S = 0$  (see Appendix D),  $10N + 14$  for the nonlinear bath with  $\epsilon_S \neq 0$  and  $3N + 2$  for the symmetric spin boson system. We have used up to  $N = 10^4$  bath oscillators. For the particular choice  $\epsilon_S = 0$ , the correlation function at discrete frequencies is given by  $K_{zz}^S(\omega_k) = (\mu_k^{11})^2/\delta\omega$ , for the generator  $\tilde{\eta}$ . All the  $h^{\alpha\beta}$  coefficients vanish for increasing flow  $l \rightarrow \infty$ , as expected. For the other generator  $\eta$ , we have to evaluate

$$K_{zz}^S(\omega) = \sum_n |\langle 0 | (h^{03}\Sigma_{03} + h^{13}\Sigma_{13}) | n \rangle|^2 \delta(\omega + E_0 - E_n) + \sum_{k,n} (\mu_k^{11})^2 |\langle 0 | \Sigma_{11} | n \rangle|^2 \delta(\omega + E_0 - E_n - \omega_k), \quad (5.1)$$

where  $E_n$  are the eigenvalues of  $H_S$  and  $|n\rangle$  are the eigenvectors of  $H_S$  for  $l \rightarrow \infty$ . In the parameter regime considered below, it turns out that the equilibrium correlator is given by the same expression as for the generator  $\tilde{\eta}$  with high accuracy.

If we choose the generator  $\tilde{\eta}$  leading to the diagonal fixed-point Hamiltonian  $\tilde{H}_0$ , which is scale-independent, then, since no asymptotic scale is present, the flow equations will first decouple the high-energy modes and then the low-energy modes. The same feature is found for

the correlation functions, see Fig. 2. They are first determined for high energies and the low-energy behavior is calculated last. The frequency regions of interest are centered on the peaks which appear at the shifted transition frequencies of the four-level system. The spectral function around these resonances is determined by a stable flow away from the asymptotic regime, which must be treated in a different way<sup>17,21,30</sup>. In practice, the flow equation can only be integrated up to  $l^* \approx (\delta\omega)^{-2}$ . We find that the integration can be stopped for much lower values; the system is decoupled with high accuracy for  $l > 20$ , see again Fig. 2.

For the other choice of the generator,  $\eta$ , we find a different behavior of the flow. The flow equations first decouple the frequency regions around the peaks. The peaks contain gaps which are closed only slowly with increasing  $l$ , see Fig. 3. Therefore, in contrast to the generator  $\tilde{\eta}$  the flow equations have to be integrated up to much higher values  $l \approx 10^5$ .

For both choices of the generator there are regions which are decoupled faster than others. Let us discuss the consequences for the numerics. The coupling constants of the Hamiltonian,  $\lambda_k^{\alpha\beta}$ , and  $\kappa_k^{\alpha\beta}$  decay while the flow parameter  $l$  is varied from zero to infinity. If we take a closer look at the flow equations, we observe that the differential equations for a certain mode  $k$  become trivial as soon as the  $\lambda_k^{\alpha\beta}$  and  $\kappa_k^{\alpha\beta}$  are approximately zero, e.g. for the generator  $\tilde{\eta}$  the higher modes are first decoupled. In the end, changes take place only in a low-frequency region, see Figs. 2, 3. This is why, in practice, all the differential equations outside of this region can be replaced by trivial ones in the program. This accelerates the calculation significantly.

The correlation function is given as a bilinear form in the  $\mu_k^{\alpha\beta}$  and  $\nu_k^{\alpha\beta}$ . The coefficients  $\mu_k^{\alpha\beta}$  and  $\nu_k^{\alpha\beta}$  turn out to be zero at certain points. For the special case of the nonlinear bath with  $\epsilon_S = 0$ , only  $\mu_k^{11}$  remains, which vanishes (intersects the frequency axis) at different points  $\omega$ . This feature is contained in the flow equations and was observed in Ref. 17 for the dissipative two-state system. The correlation function, bilinear in  $\mu_k^{11}$ , therefore has unphysical zeroes, which constitute a finite-size effect, only disappearing in the thermodynamic limit.

There are different possibilities to address this problem. In Ref. 17 it was observed that for a certain value of the flow parameter, the decoupled two-state system behaved like a dissipative harmonic oscillator. Furthermore, it was shown that for the dissipative harmonic oscillator there exists a conserved quantity which could be added at a certain value of  $l$  in order to complete the correlation function. Another possible way to deal with this problem is to integrate to a large value of  $l$ , such that the higher frequencies of the right peak are decoupled, as in Fig. 2, and to continue the integration in a restricted frequency range with a higher resolution of the bath modes. Finally, only a tiny gap is left which can then be closed by hand.

Using the generator  $\tilde{\eta}$  spikes appear around this singu-

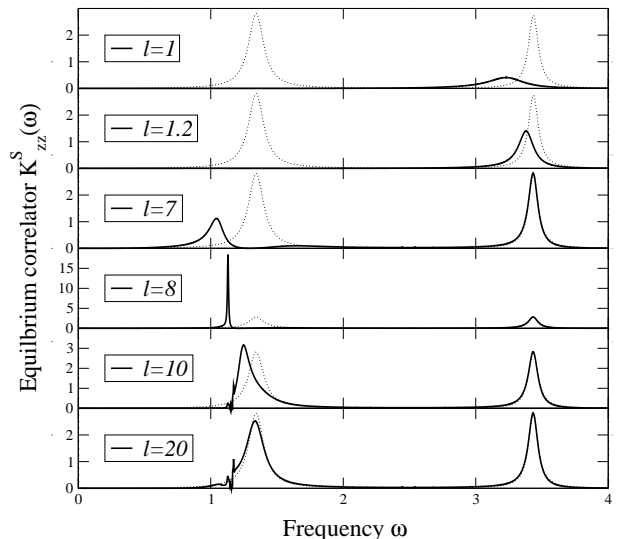


FIG. 2: Behavior of the flow using the generator  $\tilde{\eta}$ : Fourier transform of the equilibrium correlator for increasing flow parameter  $l$  (solid line), compared with the final result of a Markoff approximation (dotted line). We see that the higher modes are decoupled first, i.e., the peak on the right hand is visible already for  $l = 7$ . At  $l = 20$  the sum rule is fulfilled with an accuracy of  $\approx 0.5\%$ . This result is better than the Markoff approximation used to calculate the influence of the linear bath  $F$  on the system  $S + B$ , which should lead to good results for  $\alpha/(2\pi) = 0.01$ . Naturally, the sum rule is not valid at the beginning of the flow. The parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $g = 1$ ,  $\alpha/(2\pi) = 0.01$ , and  $N = 8000$ .

larity, which are due to the finite number of bath modes  $N$ , i.e., we assume that the phenomena will disappear for increasing  $N$ . These spikes, generated via an amplifying effect, are not completely understood. Probably, they occur always if the change in the correlation function happens too abruptly. This suggestion is motivated by the observation that the same feature appears, if the cutoff is chosen too close to the characteristic frequency of the system. There, we find that, due to the intrinsic properties of the differential-equation system, a spike appears at the cutoff frequency. The system always needs a certain frequency range to spread out smoothly.

The sum rules are fulfilled with an accuracy of  $\sim 0.5\%$  using  $\tilde{\eta}$  and  $\sim 0.1\%$  using the generator  $\eta$ , when not otherwise mentioned.

The bath cutoff was chosen to be  $\omega_c = 5$ , when not otherwise mentioned. The results depend on the choice of the bath cutoff  $\omega_c$ . However, the qualitative behavior is independent of the cutoff and remains the same for different large-enough  $\omega_c$ .

Flow equations give the best results for small coupling  $\alpha$ . This does not mean that the approach is perturbative in the usual sense, but the neglected terms in the truncation procedure are smaller in the low-coupling limit. The flow equations for our specific system obtained with the generator  $\tilde{\eta}$  were observed to give accurate results up to  $\alpha/(2\pi) = 0.1$  and the generator  $\eta$  was used up to

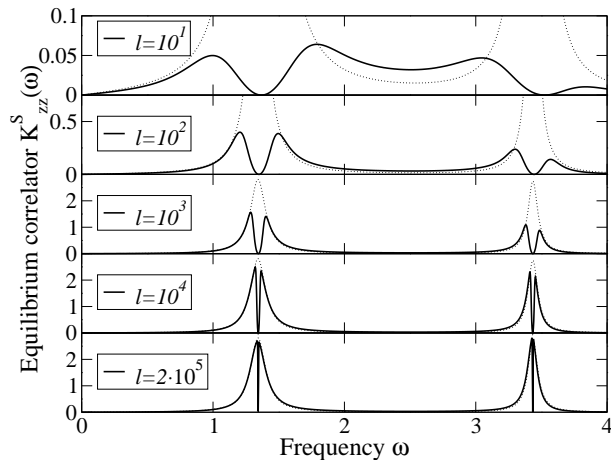


FIG. 3: Behavior of the flow using the generator  $\eta$ : Fourier transform of the equilibrium correlator for increasing flow parameter  $l$  (solid line), compared with the final result of a Markoff approximation (dotted line). We see that the frequency regions around the peaks are decoupled at last. The sum rule is not valid at the beginning of the flow, but finally it is fulfilled with an accuracy better than  $\approx 0.1\%$ . The parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $g = 1$ ,  $\alpha/(2\pi) = 0.01$ , and  $N = 2000$ .

$$\alpha/(2\pi) = 0.08.$$

## VI. COMPARISON OF NONLINEAR WITH LINEAR BATH

The relevant parameters in our model are  $\epsilon_S$ ,  $\Delta_S$ ,  $\Delta$ , the coupling strength  $g$ , and the coupling strength  $\alpha$  to the linear bath  $F$ . We choose the time scale such that  $\Delta \equiv 1$ . Furthermore, the results discussed in the following have been calculated for  $\Delta_S = 1.2$ , close to 1, to keep the decay strong without being in resonance. We set the temperature to zero ( $T = 0$ ) and vary the two parameters  $\alpha$  and  $g$ .

To begin our discussion, we note some generic features of the results obtained for the two approaches: linear and nonlinear bath. Since  $K_{zz}^S(\omega)$  is essentially the Fourier transform of the relaxational dynamics, it consists of several “Lorentzian like” peaks. Their number is constrained to be less than the maximum number of 6 transition frequencies for  $S + B$  in the case of the nonlinear-bath system. In practice, degeneracies between transition frequencies and selection rules reduce that number, e.g., to 2 for the unbiased case. More peaks, than the one transition frequency expected for a two level system, are observed in the case of the linear-bath approximation. It turns out that the second peak is induced by the peaked bath spectral density. To summarize, we have found for the unbiased case two peaks for the linear and the nonlinear bath. One can associate two different decoherence times with the two peaks. Both approaches show a similar behavior for increasing coupling  $g$ , where we ob-

serve a larger separation between the two peaks. For the unbiased case  $\epsilon_S \neq 0$ , we expect at least three peaks for weak coupling  $\alpha$ . Additional possible zero-frequency delta-distribution contributions show up for the biased case, due to the definition of the correlation functions. Moreover, in contrast to the high-temperature results, no “pure” relaxation is seen in the biased case.

Let us now focus on the unbiased case, i.e.,  $\epsilon_S = 0$ . In the limit of weak coupling,  $g \rightarrow 0$ , all that remains is a broadened peak at the transition frequency  $2\Delta_S$  of system  $S$  alone. In that limit, the results for the two approaches coincide, as expected. This is visible in the topmost plots of Figs. 5 and 6, where the Fourier transform of the equilibrium correlator is shown for increasing system-bath coupling  $g$ , while  $\alpha/(2\pi) = 0.05, 0.01$  is kept constant. We expect that for small  $g$  the non-linear and the linear-bath results fall together, i.e., in this limit a weak-coupling approximation<sup>31</sup> or a Markoff approximation<sup>32</sup> are valid and one can therefore not distinguish between linear and nonlinear bath. Furthermore, the bath spectral density was chosen such that the two approaches agree for small enough coupling.

With increasing  $g$ , the peaks are broadened and shifted, and additional peaks may appear (see Fig. 5). Indeed, the most notable difference from a master equation used for coupling of  $S$  and the nonlinear bath is the appearance of a second peak around the transition frequency  $2\Delta$  of the two-level fluctuator  $B$ . In this way, the power spectrum of the bath fluctuations appears in the short-time behavior of the correlator of the system  $S$ . This behavior cannot be captured by a master equation, where only one peak is present. Here, for the parameters chosen the linear and nonlinear bath agree nicely for  $g = 0.25$ , rather to our surprise, very well up to  $g = 0.5$ . For  $g > 0.5$  the deviations become significant. Increasing  $g$  leads to a frequency shift and a change in the width of the “original” peak at  $2\Delta_S$ . These changes are due to the change in eigenfrequencies and eigenvectors of the combined system  $S + B$ . If we keep  $g = 1$  constant and compare the results for different increasing  $\alpha$ , see Fig. 4, the differences between linear and nonlinear bath depend on the ratio  $g/\alpha$ . A small  $\alpha/(2\pi) = 0.005$  or  $\alpha/(2\pi) = 0.01$  corresponds to a structured bath (Fig. 1) and the results show that the peak shape is similar for nonlinear and linear baths, but the peak position is different. We stress that for higher coupling, e.g.,  $\alpha/(2\pi) = 0.05$  and  $\alpha/(2\pi) = 0.1$ , the nonlinear bath acquires a new structure and the shape of the peak deviates significantly from the linear bath. Thus, the qualitative differences between linear and nonlinear baths are smaller for a structured bath.

Since the linear bath takes the full bath spectrum  $\langle BB \rangle_\omega$  as input, this spectrum may show up in the result for the system correlator  $K_{zz}^S(\omega)$ , as is indeed the case. Figures 5 and 6 demonstrate that this effect is independent of the  $\alpha$  values considered here, in contrast to the infinite temperature limit<sup>4</sup>, where this effect was most pronounced for small  $\gamma$ , where the bath spectrum has a

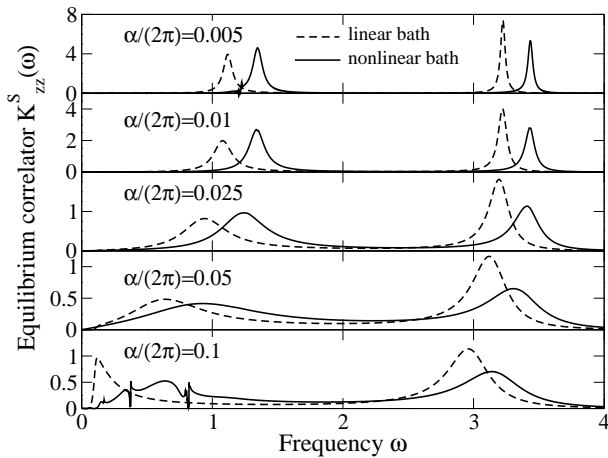


FIG. 4: Fourier transform  $K_{zz}^S(\omega)$  of the equilibrium correlator of  $\hat{\sigma}_z^S(t)$ , for different structured nonlinear baths with  $\alpha/(2\pi) = 0.005, 0.01, 0.025, 0.05$ , and  $0.1$  from topmost to lowest graph. The values of the other parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $g = 1$ , and  $\epsilon_S = 0$ . Nonlinear bath: solid line, linear bath: dashed line. Dotted line: Markoff approximation. In Sec. V we discuss the singularities appearing in the correlators, e.g., in the lowest graph at frequency  $\omega \approx 0.8$  and the topmost graph. These two graphs were obtained using the generator  $\tilde{\eta}$ ,  $N = 5000$ , and  $l = 500$ . The rest of the graphs were calculated with the generator  $\eta$ ,  $N = 2000$ , and  $l = 2 \cdot 10^5$  maximal.

relatively sharp structure. To emphasize this point, we note that even the asymmetric shape of the bath spectral density  $\langle BB \rangle_\omega$  is mapped to the equilibrium correlator, see, e.g., Fig. 5 compared with Fig. 1. This is an effect of the non-ohmic structure of the bath, which could not be observed for an ohmic linear bath. It can further only be observed when  $S$  and  $B$  are in resonance, i.e. for low frequencies the bath spectral density behaves ohmically. The second peak, which appears for higher coupling  $g$ , is therefore determined by the bath correlation function  $\langle BB \rangle_\omega$  with one peak in the vicinity of the system transition frequency and another given by the energy scale  $2\Delta_S = 2.4$ . For increasing system-bath coupling  $g$ , both peaks are shifted away from each other. Within the linear-bath approximation, the peak around  $2\Delta_S$  is shifted to higher frequencies to a lesser extent and the second peak around  $2\Delta$  is shifted to lower frequencies, more pronounced compared to the nonlinear bath.

The results of the linear and nonlinear bath agree well up to intermediate coupling strengths, as discussed previously. Nevertheless, there are discrepancies: In particular, the shifts are different. In contrast to infinite temperature, varying the parameters  $g$  and  $\alpha$  leads to a visible shift of the peaks for the linear bath. Furthermore, the peaks become wider and more asymmetric, but not as pronounced as the peak around  $2\Delta_S$  for infinite temperature. For higher values of  $\alpha$ , the linear bath, in general, shows less structure than the actual nonlinear bath, however, the asymmetric shape of the peaks is mimicked to some extent.

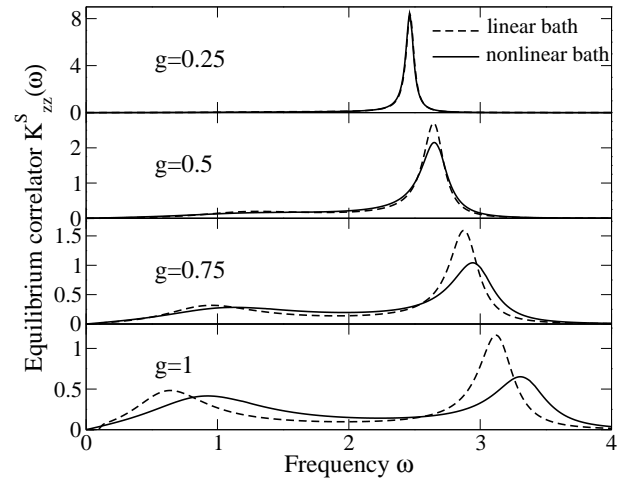


FIG. 5: Fourier transform  $K_{zz}^S(\omega)$  of the equilibrium correlator of  $\hat{\sigma}_z^S(t)$ , for different values of the system-bath coupling  $g = 0.25, 0.5, 0.75$ , and  $1$  from topmost to lowest graph. The values of the other parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $\alpha/(2\pi) = 0.1$ , and  $\epsilon_S = 0$ . Solid line: nonlinear bath,  $N = 2000$ ,  $l = 8 \cdot 10^4$  maximal, and the generator  $\eta$  was used. Dashed line: linear bath,  $N = 2000$ ,  $l = 1000$ , and the generator  $\tilde{\eta}$  was used.

Finally, to close this section let us turn to the biased case  $\epsilon_S \neq 0$ . Figure 7 shows the equilibrium correlator of  $\hat{\sigma}_z^S(t)$  for two different values of the bias:  $\epsilon_S = 0.5$  and  $\epsilon_S = 1$ . For comparison, the Markoff approximation is shown, which should lead to reasonable results for  $\alpha/(2\pi) = 0.01$ , but the agreement between the two approaches is worse than for the unbiased case. We note that the flow equations do not describe the third peak. What is the reason for this? Flow equations are equivalent to a unitary basis change. Therefore, we expect the same results for each unitarily equivalent Hamiltonian. As soon as approximations are made, things do change, since, depending on the representation of the Hamiltonian, different terms have different significance. Our approximation scheme leads to better results for the unbiased case. This was also seen in Ref. 21 for the spin-boson problem, where shifts of the bosonic modes were introduced and tuned to an optimal point in order to deal with this problem. We leave such an analysis for the dissipative four-state system for future work.

## VII. CONCLUSIONS

The first main result of this work is the derivation of flow equations for a general dissipative four-level system. With this framework, not only the correlation functions of our nonlinear bath model can be studied, but correlation functions for any four-state system coupled linearly to an oscillator bath. An additional key aspect is that there is no restriction for this bath to have an ohmic bath spectral density.

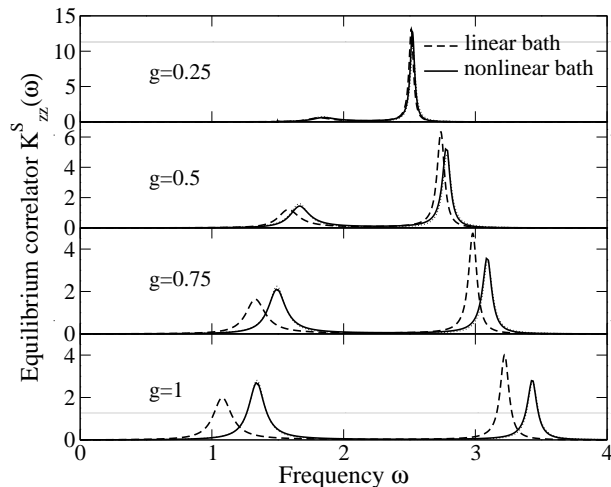


FIG. 6: Fourier transform  $K_{zz}^S(\omega)$  of the equilibrium correlator of  $\hat{\sigma}_z^S(t)$ , for different values of the system-bath coupling  $g = 0.25, 0.5, 0.75$ , and  $1$  from topmost to lowest graph. The values of the other parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $\alpha/(2\pi) = 0.01$ , and  $\epsilon_S = 0$ . Solid line: nonlinear bath,  $N = 2000$ ,  $l = 2 \cdot 10^5$  maximal, and the generator  $\eta$  was used. Dashed line: linear bath,  $N = 2000$ ,  $l = 1000$ , and the generator  $\tilde{\eta}$  was used. Dotted line (falls together almost perfectly with the solid line): Markoff approximation used to calculate the influence of the linear bath  $F$  on the system  $S + B$ .

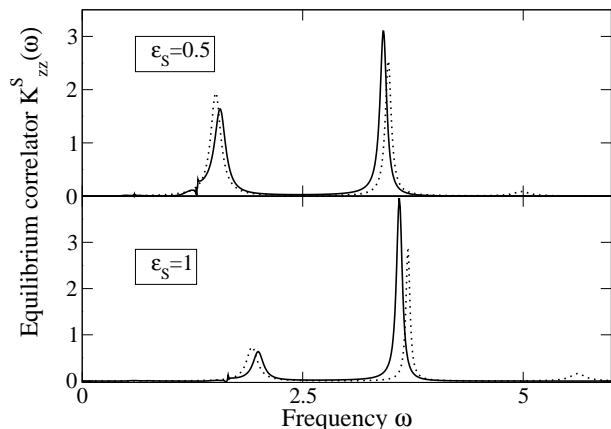


FIG. 7: Fourier transform  $K_{zz}^S(\omega)$  of the equilibrium correlator of  $\hat{\sigma}_z^S(t)$ , for two different values of the bias  $\epsilon_S = 0.5$  topmost  $\epsilon_S = 1$  lowest graph. The values of the other parameters are:  $\Delta = 1$ ,  $\Delta_S = 1.2$ ,  $\alpha/(2\pi) = 0.01$ ,  $g = 1$ ,  $\omega_C = 10$ ,  $l = 100$ , and the generator  $\tilde{\eta}$  was used. For  $\epsilon_S = 0.5$ ,  $N = 2000$  bath modes were used, the height of the delta peak at zero frequency is  $0.229$  and the sum rule leads to a value  $1.048$ . For  $\epsilon_S = 1$ ,  $N = 1000$  bath modes were used, the height of the delta peak at zero frequency is  $0.544$  and the sum rule leads to a value  $1.149$ . Solid line: nonlinear bath, dotted line: Markoff approximation used to calculate the influence of the linear bath  $F$  on the system  $S + B$ .

As an application of these general flow equations, a model of a nonlinear bath was discussed, consisting of a single two-level system subject to a linear oscillator bath. Its effect on another two-level system at zero temperature has been analyzed and compared with the effect of a linear oscillator bath. Many numerical results for various special cases have been obtained and discussed. For the flow equations of the nonlinear bath, two different generators have been used and the behavior of the corresponding flows has been analyzed.

For small system-bath coupling the equilibrium correlator contains only one peak which can be obtained using a Markoff approximation. In contrast the linear bath can describe the second peak appearing for larger coupling strengths. At least two frequencies are therefore present in the time evolution of the two-state system, and furthermore, the decoherence is strongly increased if the system and bath are in resonance. As expected, the linear-bath approximation fails for the regime of large system-bath coupling. In that regime the linear bath may undoubtedly represent a good approximation to our actual nonlinear bath up to couplings on the order of half of the system energy scale. In the strong-coupling regime, on the order of the system energy scale, the agreement of the peak shapes is qualitatively better, when the bath spectrum has a strongly peaked structure. Here, discrepancies between the linear and the original nonlinear bath became clearly visible. Although we have only discussed the simplest example of a nonlinear bath, it is likely that the linear-bath approximation might lead to reasonable results in the intermediate-coupling regime, also for other, possibly more sophisticated systems.

## Acknowledgments

We would like to thank S. Kehrein for many crucial hints and illuminating discussions, and for reading the manuscript. We would also like to acknowledge S. Gassmann and F. Marquardt. Our work was supported by the SKORE-A program, the Swiss NSF, and the NCCR Nanoscience.



# APPENDIX A: THE FLOW EQUATIONS FOR THE HAMILTONIAN

Below we list the result for the dissipative four-state system discussed in Sec. III. A sum over  $i, j, l, p, s, r$ , and  $t$  from one to three is suppressed in the formulas.

$$\begin{aligned}
\frac{d\Delta^{00}}{dl} &= 4 \sum_k \left\{ -\frac{1}{2} \left( \lambda_k^{00} \lambda_k^{00} + \lambda_k^{ij} \lambda_k^{ij} + \lambda_k^{0j} \lambda_k^{0j} + \lambda_k^{i0} \lambda_k^{i0} + \kappa_k^{00} \kappa_k^{00} + \kappa_k^{ij} \kappa_k^{ij} + \kappa_k^{0j} \kappa_k^{0j} + \kappa_k^{i0} \kappa_k^{i0} \right) \omega_k \right. \\
&\quad - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \epsilon_{plj} \lambda_k^{0j} - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \epsilon_{sli} \lambda_k^{i0} - (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \lambda_k^{ij} \\
&\quad \left. + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \epsilon_{plj} \kappa_k^{0j} + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \epsilon_{sli} \kappa_k^{i0} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \kappa_k^{ij} \right\} \\
\frac{d\Delta^{0n}}{dl} &= 4 \sum_k \left\{ \left[ -\lambda_k^{i0} \lambda_k^{in} - \lambda_k^{00} \lambda_k^{0n} - \kappa_k^{i0} \kappa_k^{in} - \kappa_k^{00} \kappa_k^{0n} + \frac{1}{2} (\kappa_k^{0i} \lambda_k^{0l} + \kappa_k^{ji} \lambda_k^{jl} - \lambda_k^{0i} \kappa_k^{0l} - \lambda_k^{ji} \kappa_k^{jl}) \epsilon_{iln} (2n_k + 1) \right] \omega_k \right. \\
&\quad - [(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \lambda_k^{0r} \epsilon_{plj} \epsilon_{jrn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \lambda_k^{ir} \epsilon_{jrn} \\
&\quad + (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \kappa_k^{0r} \epsilon_{plj} \epsilon_{jrn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \kappa_k^{ir} \epsilon_{jrn}] (2n_k + 1) \\
&\quad - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \lambda_k^{00} \epsilon_{pln} - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \lambda_k^{in} \epsilon_{sli} - (\Delta^{0p} \kappa_k^{il} \epsilon_{pln} + \Delta^{s0} \kappa_k^{ln} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{pln} + \Delta^{sn} \kappa_k^{l0} \epsilon_{sli}) \lambda_k^{i0} \\
&\quad \left. + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \kappa_k^{00} \epsilon_{pln} + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \kappa_k^{in} \epsilon_{sli} + (\Delta^{0p} \lambda_k^{il} \epsilon_{pln} + \Delta^{s0} \lambda_k^{ln} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{pln} + \Delta^{sn} \lambda_k^{l0} \epsilon_{sli}) \kappa_k^{i0} \right\} \\
\frac{d\Delta^{n0}}{dl} &= 4 \sum_k \left\{ \left[ -\lambda_k^{0i} \lambda_k^{ni} - \lambda_k^{00} \lambda_k^{n0} - \kappa_k^{0i} \kappa_k^{ni} - \kappa_k^{00} \kappa_k^{n0} + \frac{1}{2} (\kappa_k^{i0} \lambda_k^{l0} + \kappa_k^{ij} \lambda_k^{lj} - \lambda_k^{i0} \kappa_k^{l0} - \lambda_k^{ij} \kappa_k^{lj}) \epsilon_{iln} (2n_k + 1) \right] \omega_k \right. \\
&\quad - [(\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \lambda_k^{r0} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \lambda_k^{rj} \epsilon_{irn} \\
&\quad + (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \kappa_k^{r0} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \kappa_k^{rj} \epsilon_{irn}] (2n_k + 1) \\
&\quad - (\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) \lambda_k^{0j} - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \lambda_k^{00} \epsilon_{sln} - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \lambda_k^{nj} \epsilon_{plj} \\
&\quad \left. + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) \kappa_k^{0j} + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \kappa_k^{nj} \epsilon_{plj} + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \kappa_k^{00} \epsilon_{sln} \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{d\Delta^{nm}}{dl} &= 4 \sum_k \left\{ \left( -\lambda_k^{00} \lambda_k^{nm} - \lambda_k^{0m} \lambda_k^{n0} - \kappa_k^{00} \kappa_k^{nm} - \kappa_k^{0m} \kappa_k^{n0} + \frac{1}{2} (\lambda_k^{ij} \lambda_k^{pl} + \kappa_k^{ij} \kappa_k^{pl}) \epsilon_{ipn} \epsilon_{jlm} \right. \right. \\
&\quad \left. + \frac{1}{2} [(\kappa_k^{i0} \lambda_k^{lm} + \kappa_k^{im} \lambda_k^{l0} - \lambda_k^{i0} \kappa_k^{lm} - \lambda_k^{im} \kappa_k^{l0}) \epsilon_{iln} + (\kappa_k^{0i} \lambda_k^{nl} + \kappa_k^{ni} \lambda_k^{0l} - \lambda_k^{0i} \kappa_k^{nl} - \lambda_k^{ni} \kappa_k^{0l}) \epsilon_{ilm}] (2n_k + 1) \right) \omega_k \\
&\quad - [(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \lambda_k^{nr} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) \lambda_k^{0r} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \lambda_k^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sli}) \lambda_k^{r0} \epsilon_{irn} \\
&\quad + (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \kappa_k^{nr} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) \kappa_k^{0r} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \kappa_k^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sli}) \kappa_k^{r0} \epsilon_{irn}] (2n_k + 1) \\
&\quad - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \lambda_k^{n0} \epsilon_{plm} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \lambda_k^{tr} \epsilon_{itn} \epsilon_{jrm} \\
&\quad - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \lambda_k^{0m} \epsilon_{sln} - (\Delta^{0p} \kappa_k^{nl} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sln}) \lambda_k^{00} \\
&\quad + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \kappa_k^{n0} \epsilon_{plm} - (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \kappa_k^{tr} \epsilon_{itn} \epsilon_{jrm} \\
&\quad \left. + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \kappa_k^{0m} \epsilon_{sln} + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sln}) \kappa_k^{00} \right\}
\end{aligned}$$

(A1)

$$\begin{aligned}
\frac{d\lambda_k^{00}}{dl} &= -\omega_k^2 \lambda_k^{00} + 2 \sum_q \eta_{kq} \lambda_q^{00} \\
\frac{d\lambda_k^{0n}}{dl} &= -\omega_k^2 \lambda_k^{0n} + 4(\kappa_k^{0i} \Delta^{0l} + \kappa_k^{ji} \Delta^{jl}) \epsilon_{iln} \omega_k + 2 \sum_q \eta_{kq} \lambda_q^{0n} \\
&\quad - 4(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \Delta^{0r} \epsilon_{plj} \epsilon_{jrn} - 4(\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \Delta^{ir} \epsilon_{jrn} \\
\frac{d\lambda_k^{n0}}{dl} &= -\omega_k^2 \lambda_k^{n0} + 4(\kappa_k^{i0} \Delta^{l0} + \kappa_k^{ij} \Delta^{lj}) \epsilon_{iln} \omega_k + 2 \sum_q \eta_{kq} \lambda_q^{n0} \\
&\quad - 4(\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \Delta^{r0} \epsilon_{sli} \epsilon_{irn} - 4(\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \Delta^{rj} \epsilon_{irn} \\
\frac{d\lambda_k^{nm}}{dl} &= -\omega_k^2 \lambda_k^{nm} + 4\left((\kappa_k^{0i} \Delta^{nl} + \kappa_k^{ni} \Delta^{0l}) \epsilon_{ilm} + (\kappa_k^{i0} \Delta^{lm} + \kappa_k^{im} \Delta^{l0}) \epsilon_{iln}\right) \omega_k + 2 \sum_q \eta_{kq} \lambda_q^{nm} \\
&\quad - 4(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \Delta^{nr} \epsilon_{plj} \epsilon_{jrm} - 4(\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) \Delta^{0r} \epsilon_{jrm} \\
&\quad - 4(\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \Delta^{rm} \epsilon_{sli} \epsilon_{irn} - 4(\Delta^{0p} \lambda_k^{il} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sli}) \Delta^{r0} \epsilon_{irn} \\
\\
\frac{d\kappa_k^{00}}{dl} &= -\omega_k^2 \kappa_k^{00} - 2 \sum_q \eta_{qk} \kappa_q^{00} \\
\frac{d\kappa_k^{0n}}{dl} &= -\omega_k^2 \kappa_k^{0n} - 4(\lambda_k^{0i} \Delta^{0l} + \lambda_k^{ji} \Delta^{jl}) \epsilon_{iln} \omega_k - 2 \sum_q \eta_{qk} \kappa_q^{0n} \\
&\quad - 4(\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \Delta^{0r} \epsilon_{plj} \epsilon_{jrn} - 4(\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \Delta^{ir} \epsilon_{jrn} \\
\frac{d\kappa_k^{n0}}{dl} &= -\omega_k^2 \kappa_k^{n0} - 4(\lambda_k^{i0} \Delta^{l0} + \lambda_k^{ij} \Delta^{lj}) \epsilon_{iln} \omega_k - 2 \sum_q \eta_{qk} \kappa_q^{n0} \\
&\quad - 4(\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \Delta^{r0} \epsilon_{sli} \epsilon_{irn} - 4(\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \Delta^{rj} \epsilon_{irn} \\
\frac{d\kappa_k^{nm}}{dl} &= -\omega_k^2 \kappa_k^{nm} - 4\left((\lambda_k^{0i} \Delta^{nl} + \lambda_k^{ni} \Delta^{0l}) \epsilon_{ilm} + (\lambda_k^{i0} \Delta^{lm} + \lambda_k^{im} \Delta^{l0}) \epsilon_{iln}\right) \omega_k - 2 \sum_q \eta_{qk} \kappa_q^{nm} \\
&\quad - 4(\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \Delta^{nr} \epsilon_{plj} \epsilon_{jrm} - 4(\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) \Delta^{0r} \epsilon_{jrm} \\
&\quad - 4(\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \Delta^{rm} \epsilon_{sli} \epsilon_{irn} - 4(\Delta^{0p} \kappa_k^{il} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sli}) \Delta^{r0} \epsilon_{irn}
\end{aligned}$$

## APPENDIX B: THE $\eta_{kq}$ COEFFICIENTS

$$\begin{aligned}
\eta_{kq} &= \frac{1}{\omega_q^2 - \omega_k^2} \left\{ 2\omega_k \left[ (\lambda_k^{0i} \kappa_q^{nl} \epsilon_{ilm} \langle \Sigma_{nm} \rangle + \lambda_k^{i0} \kappa_q^{lm} \epsilon_{iln} \langle \Sigma_{nm} \rangle + \lambda_k^{nj} \kappa_q^{0l} \epsilon_{jlm} \langle \Sigma_{nm} \rangle + \lambda_k^{im} \kappa_q^{l0} \epsilon_{iln} \langle \Sigma_{nm} \rangle \right. \right. \\
&\quad + \lambda_k^{0i} \kappa_q^{0l} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \lambda_k^{i0} \kappa_q^{l0} \epsilon_{iln} \langle \Sigma_{n0} \rangle + \lambda_k^{ji} \kappa_q^{jl} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \lambda_k^{ij} \kappa_q^{lj} \epsilon_{iln} \langle \Sigma_{n0} \rangle) \omega_k \\
&\quad + (\lambda_q^{0i} \kappa_k^{nl} \epsilon_{ilm} \langle \Sigma_{nm} \rangle + \lambda_q^{i0} \kappa_k^{lm} \epsilon_{iln} \langle \Sigma_{nm} \rangle + \lambda_q^{nj} \kappa_k^{0l} \epsilon_{jlm} \langle \Sigma_{nm} \rangle + \lambda_q^{im} \kappa_k^{l0} \epsilon_{iln} \langle \Sigma_{nm} \rangle \\
&\quad + \lambda_q^{0i} \kappa_k^{0l} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \lambda_q^{i0} \kappa_k^{l0} \epsilon_{iln} \langle \Sigma_{n0} \rangle + \lambda_q^{ji} \kappa_k^{jl} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \lambda_q^{ij} \kappa_k^{lj} \epsilon_{iln} \langle \Sigma_{n0} \rangle) \omega_q \Big] \\
&\quad - 2\omega_q \left[ (\kappa_k^{0i} \lambda_q^{nl} \epsilon_{ilm} \langle \Sigma_{nm} \rangle + \kappa_k^{i0} \lambda_q^{lm} \epsilon_{iln} \langle \Sigma_{nm} \rangle + \kappa_k^{nj} \lambda_q^{0l} \epsilon_{jlm} \langle \Sigma_{nm} \rangle + \kappa_k^{im} \lambda_q^{l0} \epsilon_{iln} \langle \Sigma_{nm} \rangle \right. \\
&\quad + \kappa_k^{0i} \lambda_q^{0l} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \kappa_k^{i0} \lambda_q^{l0} \epsilon_{iln} \langle \Sigma_{n0} \rangle + \kappa_k^{ji} \lambda_q^{jl} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \kappa_k^{ij} \lambda_q^{lj} \epsilon_{iln} \langle \Sigma_{n0} \rangle) \omega_k \\
&\quad + (\kappa_q^{0i} \lambda_k^{nl} \epsilon_{ilm} \langle \Sigma_{nm} \rangle + \kappa_q^{i0} \lambda_k^{lm} \epsilon_{iln} \langle \Sigma_{nm} \rangle + \kappa_q^{nj} \lambda_k^{0l} \epsilon_{jlm} \langle \Sigma_{nm} \rangle + \kappa_q^{im} \lambda_k^{l0} \epsilon_{iln} \langle \Sigma_{nm} \rangle \\
&\quad + \kappa_q^{0i} \lambda_k^{0l} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \kappa_q^{i0} \lambda_k^{l0} \epsilon_{iln} \langle \Sigma_{n0} \rangle + \kappa_q^{ji} \lambda_k^{jl} \epsilon_{iln} \langle \Sigma_{0n} \rangle + \kappa_q^{ij} \lambda_k^{lj} \epsilon_{iln} \langle \Sigma_{n0} \rangle) \omega_q \Big]
\end{aligned} \tag{B1}$$

$$\begin{aligned}
& + 4\omega_k \left[ (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\kappa_q^{nr}\epsilon_{plj}\epsilon_{jrm}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\kappa_k^{nl}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sln} + \Delta^{np}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sln})\kappa_q^{0r}\epsilon_{jrm}\langle\Sigma_{nm}\rangle \right. \\
& + (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\kappa_q^{rm}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plm} + \Delta^{s0}\kappa_k^{lm}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plm} + \Delta^{sm}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{r0}\epsilon_{irn}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\kappa_q^{0r}\epsilon_{plj}\epsilon_{jrn}\langle\Sigma_{0n}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{ir}\epsilon_{jrn}\langle\Sigma_{0n}\rangle \\
& + (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\kappa_q^{r0}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{n0}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{rj}\epsilon_{irn}\langle\Sigma_{n0}\rangle \\
& + (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\kappa_q^{nr}\epsilon_{plj}\epsilon_{jrm}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\kappa_k^{nl}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sln} + \Delta^{np}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sln})\kappa_q^{0r}\epsilon_{jrm}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\kappa_q^{rm}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plm} + \Delta^{s0}\kappa_k^{lm}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plm} + \Delta^{sm}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{r0}\epsilon_{irn}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\kappa_q^{0r}\epsilon_{plj}\epsilon_{jrn}\langle\Sigma_{0n}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{ir}\epsilon_{jrn}\langle\Sigma_{0n}\rangle \\
& + (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\kappa_q^{r0}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{n0}\rangle + (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\kappa_q^{rj}\epsilon_{irn}\langle\Sigma_{n0}\rangle \Big] \\
& + 4\omega_q \left[ (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\lambda_q^{nr}\epsilon_{plj}\epsilon_{jrm}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\lambda_k^{nl}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sln} + \Delta^{np}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sln})\lambda_q^{0r}\epsilon_{jrm}\langle\Sigma_{nm}\rangle \right. \\
& + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\lambda_q^{rm}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plm} + \Delta^{s0}\lambda_k^{lm}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plm} + \Delta^{sm}\lambda_k^{l0}\epsilon_{sli})\lambda_q^{r0}\epsilon_{irn}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\lambda_q^{0r}\epsilon_{plj}\epsilon_{jrn}\langle\Sigma_{0n}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\lambda_q^{ir}\epsilon_{jrn}\langle\Sigma_{0n}\rangle \\
& + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\lambda_q^{r0}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{n0}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\lambda_q^{rj}\epsilon_{irn}\langle\Sigma_{n0}\rangle \\
& + (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\lambda_k^{nr}\epsilon_{plj}\epsilon_{jrm}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\lambda_k^{nl}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sln} + \Delta^{np}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sln})\lambda_k^{0r}\epsilon_{jrm}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\lambda_k^{rm}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{nm}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plm} + \Delta^{s0}\lambda_k^{lm}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plm} + \Delta^{sm}\lambda_k^{l0}\epsilon_{sli})\lambda_k^{r0}\epsilon_{irn}\langle\Sigma_{nm}\rangle \\
& + (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\lambda_k^{0r}\epsilon_{plj}\epsilon_{jrn}\langle\Sigma_{0n}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\lambda_k^{ir}\epsilon_{jrn}\langle\Sigma_{0n}\rangle \\
& + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\lambda_k^{r0}\epsilon_{sli}\epsilon_{irn}\langle\Sigma_{n0}\rangle + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\lambda_k^{rj}\epsilon_{irn}\langle\Sigma_{n0}\rangle \Big] \Big\}.
\end{aligned}$$

The expectation values of  $\Sigma_{\alpha\beta}$  are defined by  $\langle\Sigma_{\alpha\beta}\rangle = \text{tr}\{\Sigma_{\alpha\beta}\exp(-\beta H_{SB})\}/\text{tr}\{\exp(-\beta H_{SB})\}$ . Here,  $H_{SB}$  is the Hamiltonian of the four-level system.

### APPENDIX C: THE FLOW EQUATIONS FOR THE OBSERVABLES

$$\begin{aligned}
\frac{dh^{00}}{dl} &= 4 \sum_k \left\{ -\frac{1}{2} \left( \lambda_k^{0i}\mu_k^{0i} + \lambda_k^{i0}\mu_k^{i0} + \lambda_k^{00}\mu_k^{00} + \lambda_k^{ij}\mu_k^{ij} + \kappa_k^{0i}\nu_k^{0i} + \kappa_k^{i0}\nu_k^{i0} + \kappa_k^{00}\nu_k^{00} + \kappa_k^{ij}\nu_k^{ij} \right) \omega_k \right. \\
&\quad - (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\mu_k^{0j}\epsilon_{plj} - (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\mu_k^{i0}\epsilon_{sli} - (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\mu_k^{ij} \\
&\quad + (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\nu_k^{0j}\epsilon_{plj} + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\nu_k^{i0}\epsilon_{sli} + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\nu_k^{ij} \Big\} \\
\frac{dh^{0n}}{dl} &= 4 \sum_k \left\{ \frac{1}{2} \left( -\lambda_k^{i0}\mu_k^{in} - \lambda_k^{00}\mu_k^{0n} - \lambda_k^{in}\mu_k^{i0} - \lambda_k^{0n}\mu_k^{00} - \kappa_k^{i0}\nu_k^{in} - \kappa_k^{00}\nu_k^{0n} - \kappa_k^{in}\nu_k^{i0} - \kappa_k^{0n}\nu_k^{00} \right. \right. \\
&\quad + [\kappa_k^{0i}\mu_k^{0l} + \kappa_k^{ji}\mu_k^{jl} - \lambda_k^{0i}\nu_k^{0l} - \lambda_k^{ji}\nu_k^{jl}]\epsilon_{iln}(2n+1) \Big) \omega_k \\
&\quad - [(\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\mu_k^{0r}\epsilon_{plj}\epsilon_{jrn} + (\Delta^{0p}\lambda_k^{il}\epsilon_{plj} + \Delta^{s0}\lambda_k^{lj}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{plj} + \Delta^{sj}\lambda_k^{l0}\epsilon_{sli})\mu_k^{ir}\epsilon_{jrn} \\
&\quad + (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\nu_k^{0r}\epsilon_{plj}\epsilon_{jrn} + (\Delta^{0p}\kappa_k^{il}\epsilon_{plj} + \Delta^{s0}\kappa_k^{lj}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{plj} + \Delta^{sj}\kappa_k^{l0}\epsilon_{sli})\nu_k^{ir}\epsilon_{jrn}] (2n+1) \\
&\quad - (\Delta^{0p}\kappa_k^{0l} + \Delta^{sp}\kappa_k^{sl})\mu_k^{00}\epsilon_{pln} - (\Delta^{s0}\kappa_k^{l0} + \Delta^{sp}\kappa_k^{lp})\mu_k^{in}\epsilon_{sli} - (\Delta^{0p}\kappa_k^{il}\epsilon_{pln} + \Delta^{s0}\kappa_k^{ln}\epsilon_{sli} + \Delta^{ip}\kappa_k^{0l}\epsilon_{pln} + \Delta^{sn}\kappa_k^{l0}\epsilon_{sli})\mu_k^{i0} \\
&\quad + (\Delta^{sp}\lambda_k^{sl} + \Delta^{0p}\lambda_k^{0l})\nu_k^{00}\epsilon_{pln} + (\Delta^{0p}\lambda_k^{il}\epsilon_{pln} + \Delta^{s0}\lambda_k^{ln}\epsilon_{sli} + \Delta^{ip}\lambda_k^{0l}\epsilon_{pln} + \Delta^{sn}\lambda_k^{l0}\epsilon_{sli})\nu_k^{i0} + (\Delta^{s0}\lambda_k^{l0} + \Delta^{sp}\lambda_k^{lp})\nu_k^{in}\epsilon_{sli} \Big\}
\end{aligned}$$

$$\begin{aligned}
\frac{dh^{n0}}{dl} &= 4 \sum_k \left\{ \frac{1}{2} \left( -\lambda_k^{0i} \mu_k^{ni} - \lambda_k^{00} \mu_k^{n0} - \lambda_k^{ni} \mu_k^{0i} - \lambda_k^{n0} \mu_k^{00} - \kappa_k^{0i} \nu_k^{ni} - \kappa_k^{00} \nu_k^{n0} - \kappa_k^{ni} \nu_k^{0i} - \kappa_k^{n0} \nu_k^{00} \right. \right. \\
&\quad + \left. \left[ \kappa_k^{i0} \mu_k^{l0} + \kappa_k^{ij} \mu_k^{lj} - \lambda_k^{i0} \nu_k^{l0} - \lambda_k^{ij} \nu_k^{lj} \right] \epsilon_{iln} (2n_k + 1) \right) \omega_k \\
&\quad - \left[ (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \mu_k^{rj} \epsilon_{irn} + (\Delta^{s0} \lambda_k^{l0} \epsilon_{sli} + \Delta^{sp} \lambda_k^{lp} \epsilon_{sli}) \mu_k^{r0} \epsilon_{irn} \right. \\
&\quad + (\Delta^{s0} \kappa_k^{l0} \epsilon_{sli} + \Delta^{sp} \kappa_k^{lp} \epsilon_{sli}) \nu_k^{r0} \epsilon_{irn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \nu_k^{rj} \epsilon_{irn} \left. \right] (2n_k + 1) \\
&\quad - (\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) \mu_k^{0j} - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \mu_k^{00} \epsilon_{sln} - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \mu_k^{nj} \epsilon_{plj} \\
&\quad + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) \nu_k^{0j} + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \nu_k^{nj} \epsilon_{plj} + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \nu_k^{00} \epsilon_{sln} \left. \right\} \\
\frac{dh^{nm}}{dl} &= 4 \sum_k \left\{ \frac{1}{2} \left( -\lambda_k^{00} \mu_k^{nm} - \lambda_k^{0m} \mu_k^{n0} - \lambda_k^{n0} \mu_k^{0m} - \lambda_k^{nm} \mu_k^{00} - \kappa_k^{00} \nu_k^{nm} - \kappa_k^{0m} \nu_k^{n0} - \kappa_k^{n0} \nu_k^{0m} - \kappa_k^{nm} \nu_k^{00} \right. \right. \\
&\quad + \left. \left( \lambda_k^{ij} \mu_k^{pl} + \kappa_k^{ij} \nu_k^{pl} \right) \epsilon_{ipn} \epsilon_{jlm} \right. \\
&\quad + \left. \left[ (\kappa_k^{i0} \mu_k^{lm} + \kappa_k^{im} \mu_k^{l0} - \lambda_k^{i0} \nu_k^{lm} - \lambda_k^{im} \nu_k^{l0}) \epsilon_{iln} + (\kappa_k^{0i} \mu_k^{nl} + \kappa_k^{ni} \mu_k^{0l} - \lambda_k^{0i} \nu_k^{nl} - \lambda_k^{ni} \nu_k^{0l}) \epsilon_{ilm} \right] (2n_k + 1) \right) \omega_k \\
&\quad - \left[ (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \mu_k^{nr} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) \mu_k^{0r} \epsilon_{jrm} \right. \\
&\quad + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \mu_k^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sli}) \mu_k^{r0} \epsilon_{irn} \\
&\quad + (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \nu_k^{rm} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) \nu_k^{0r} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \nu_k^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sli}) \nu_k^{r0} \epsilon_{irn} \left. \right] (2n_k + 1) \\
&\quad - (\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) \mu_k^{n0} \epsilon_{plm} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) \mu_k^{tr} \epsilon_{itn} \epsilon_{jrm} \\
&\quad - (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) \mu_k^{0m} \epsilon_{sln} - (\Delta^{0p} \kappa_k^{nl} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sln}) \mu_k^{00} \\
&\quad + (\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) \nu_k^{n0} \epsilon_{plm} - (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) \nu_k^{tr} \epsilon_{itn} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) \nu_k^{0m} \epsilon_{sln} + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sln}) \nu_k^{00} \left. \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{d\mu_k^{00}}{dl} &= 2 \sum_q \eta_{kq} \mu_q^{00} \\
\frac{d\mu_k^{0n}}{dl} &= 2(\kappa_k^{0i} h^{0l} + \kappa_k^{ji} h^{jl}) \epsilon_{iln} \omega_k + 2 \sum_q \eta_{kq} \mu_q^{0n} \\
&\quad - 4[(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) h^{0r} \epsilon_{plj} \epsilon_{jrn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) h^{ir} \epsilon_{jrn}] \\
\frac{d\mu_k^{n0}}{dl} &= 2(\kappa_k^{i0} h^{l0} + \kappa_k^{ij} h^{lj}) \epsilon_{iln} \omega_k + 2 \sum_q \eta_{kq} \mu_q^{n0} \\
&\quad - 4[(\Delta^{0p} \lambda_k^{il} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sli}) h^{rj} \epsilon_{irn} + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) h^{r0} \epsilon_{sli} \epsilon_{irn}] \\
\frac{d\mu_k^{nm}}{dl} &= 2 \left( (\kappa_k^{0i} h^{nl} + \kappa_k^{ni} h^{0l}) \epsilon_{ilm} + (\kappa_k^{i0} h^{lm} + \kappa_k^{im} h^{l0}) \epsilon_{ilm} \right) \omega_k + 2 \sum_q \eta_{kq} \mu_q^{nm} \\
&\quad - 4[(\Delta^{sp} \lambda_k^{sl} + \Delta^{0p} \lambda_k^{0l}) h^{nr} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \lambda_k^{nl} \epsilon_{plj} + \Delta^{s0} \lambda_k^{lj} \epsilon_{sln} + \Delta^{np} \lambda_k^{0l} \epsilon_{plj} + \Delta^{sj} \lambda_k^{l0} \epsilon_{sln}) h^{0r} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \lambda_k^{l0} + \Delta^{sp} \lambda_k^{lp}) h^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \lambda_k^{il} \epsilon_{plm} + \Delta^{s0} \lambda_k^{lm} \epsilon_{sli} + \Delta^{ip} \lambda_k^{0l} \epsilon_{plm} + \Delta^{sm} \lambda_k^{l0} \epsilon_{sli}) h^{r0} \epsilon_{irn}]
\end{aligned} \tag{C1}$$

$$\begin{aligned}
\frac{d\nu_k^{00}}{dl} &= -2 \sum_q \eta_{qk} \nu_q^{00} \\
\frac{d\nu_k^{0n}}{dl} &= -2(\lambda_k^{0i} h^{0l} + \lambda_k^{ji} h^{jl}) \epsilon_{iln} \omega_k - 2 \sum_q \eta_{qk} \nu_q^{0n} \\
&\quad - 4[(\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) h^{0r} \epsilon_{plj} \epsilon_{jrn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) h^{ir} \epsilon_{jrn}] \\
\frac{d\nu_k^{n0}}{dl} &= -2(\lambda_k^{i0} h^{l0} + \lambda_k^{ij} h^{lj}) \epsilon_{iln} \omega_k - 2 \sum_q \eta_{qk} \nu_q^{n0} \\
&\quad - 4[(\Delta^{0p} \kappa_k^{il} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sli}) h^{rj} \epsilon_{irn} + (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) h^{r0} \epsilon_{sli} \epsilon_{irn}] \\
\frac{d\nu_k^{nm}}{dl} &= -2\left((\lambda_k^{0i} h^{nl} + \lambda_k^{ni} h^{0l}) \epsilon_{ilm} + (\lambda_k^{i0} h^{lm} + \lambda_k^{im} h^{l0}) \epsilon_{iln}\right) \omega_k - 2 \sum_q \eta_{qk} \nu_q^{nm} \\
&\quad - 4[(\Delta^{0p} \kappa_k^{0l} + \Delta^{sp} \kappa_k^{sl}) h^{nr} \epsilon_{plj} \epsilon_{jrm} + (\Delta^{0p} \kappa_k^{nl} \epsilon_{plj} + \Delta^{s0} \kappa_k^{lj} \epsilon_{sln} + \Delta^{np} \kappa_k^{0l} \epsilon_{plj} + \Delta^{sj} \kappa_k^{l0} \epsilon_{sln}) h^{0r} \epsilon_{jrm} \\
&\quad + (\Delta^{s0} \kappa_k^{l0} + \Delta^{sp} \kappa_k^{lp}) h^{rm} \epsilon_{sli} \epsilon_{irn} + (\Delta^{0p} \kappa_k^{il} \epsilon_{plm} + \Delta^{s0} \kappa_k^{lm} \epsilon_{sli} + \Delta^{ip} \kappa_k^{0l} \epsilon_{plm} + \Delta^{sm} \kappa_k^{l0} \epsilon_{sli}) h^{r0} \epsilon_{irn}].
\end{aligned}$$

#### APPENDIX D: A SPECIAL CASE, THE BIASED SYSTEM

Here we present the flow equations for the specific case  $\epsilon_S = 0$ , which was mainly used in this work.

$$\begin{aligned}
\frac{d\Delta^{00}}{dl} &= \sum_k [-2(\lambda_k^{31} \lambda_k^{31} + \lambda_k^{03} \lambda_k^{03} + \kappa_k^{21} \kappa_k^{21} + \kappa_k^{02} \kappa_k^{02}) \omega_k \\
&\quad + 8(-\Delta^{01} \lambda_k^{03} \kappa_k^{02} + \Delta^{33} \lambda_k^{31} \kappa_k^{02} - \Delta^{10} \lambda_k^{31} \kappa_k^{21} + \Delta^{22} \lambda_k^{03} \kappa_k^{21})] \\
\frac{d\Delta^{01}}{dl} &= \sum_k 4(\kappa_k^{02} \lambda_k^{03} \omega_k + \Delta^{01} \lambda_k^{03} \lambda_k^{03} - \Delta^{33} \lambda_k^{31} \lambda_k^{03} + \Delta^{01} \kappa_k^{02} \kappa_k^{02} - \Delta^{22} \kappa_k^{21} \kappa_k^{02}) \\
\frac{d\Delta^{10}}{dl} &= \sum_k 4(\kappa_k^{21} \lambda_k^{31} \omega_k + \Delta^{10} \lambda_k^{31} \lambda_k^{31} - \Delta^{22} \lambda_k^{03} \lambda_k^{31} + \Delta^{10} \kappa_k^{21} \kappa_k^{21} - \Delta^{33} \kappa_k^{02} \kappa_k^{21}) \\
\frac{d\Delta^{22}}{dl} &= \sum_k 4(-\kappa_k^{21} \lambda_k^{03} \omega_k - \Delta^{10} \lambda_k^{31} \lambda_k^{03} + \Delta^{22} \lambda_k^{03} \lambda_k^{03} - \Delta^{01} \kappa_k^{02} \kappa_k^{21} + \Delta^{22} \kappa_k^{21} \kappa_k^{21}) \\
\frac{d\Delta^{33}}{dl} &= \sum_k 4(-\kappa_k^{02} \lambda_k^{31} \omega_k - \Delta^{01} \lambda_k^{31} \lambda_k^{03} + \Delta^{33} \lambda_k^{31} \lambda_k^{31} - \Delta^{10} \kappa_k^{21} \kappa_k^{02} + \Delta^{33} \kappa_k^{02} \kappa_k^{02}) \\
\frac{d\lambda_k^{03}}{dl} &= -\omega_k^2 \lambda_k^{03} - 4\kappa_k^{02} \Delta^{01} \omega_k + 4\kappa_k^{21} \Delta^{22} \omega_k + 2 \sum_q \eta_{kq} \lambda_q^{03} \\
&\quad - 4\Delta^{01} \Delta^{01} \lambda_k^{03} + 4\Delta^{33} \Delta^{01} \lambda_k^{31} + 4\Delta^{10} \Delta^{22} \lambda_k^{31} - 4\Delta^{22} \Delta^{22} \lambda_k^{03} \\
\frac{d\lambda_k^{31}}{dl} &= -\omega_k^2 \lambda_k^{31} - 4\kappa_k^{21} \Delta^{10} \omega_k + 4\kappa_k^{02} \Delta^{33} \omega_k + 2 \sum_q \eta_{kq} \lambda_q^{31} \\
&\quad + 4\Delta^{01} \Delta^{33} \lambda_k^{03} - 4\Delta^{33} \Delta^{33} \lambda_k^{31} - 4\Delta^{10} \Delta^{10} \lambda_k^{31} + 4\Delta^{22} \Delta^{10} \lambda_k^{03}
\end{aligned}$$

$$\begin{aligned}
\frac{d\kappa_k^{02}}{dl} &= -\omega_k^2 \kappa_k^{02} - 4\lambda_k^{03} \Delta^{01} \omega_k + 4\lambda_k^{31} \Delta^{33} \omega_k - 2 \sum_q \eta_{qk} \kappa_q^{02} \\
&\quad - 4\Delta^{01} \Delta^{01} \kappa_k^{02} + 4\Delta^{22} \Delta^{01} \kappa_k^{21} + 4\Delta^{10} \Delta^{33} \kappa_k^{21} - 4\Delta^{33} \Delta^{33} \kappa_k^{02} \\
\frac{d\kappa_k^{21}}{dl} &= -\omega_k^2 \kappa_k^{21} - 4\lambda_k^{31} \Delta^{10} \omega_k + 4\lambda_k^{03} \Delta^{22} \omega_k - 2 \sum_q \eta_{qk} \kappa_q^{21} \\
&\quad + 4\Delta^{01} \Delta^{22} \kappa_k^{02} - 4\Delta^{22} \Delta^{22} \kappa_k^{21} - 4\Delta^{10} \Delta^{10} \kappa_k^{21} + 4\Delta^{33} \Delta^{10} \kappa_k^{02} \\
\frac{d\mu_k^{11}}{dl} &= 2\kappa_k^{02} h^{13} \omega_k + 2\kappa_k^{21} h^{30} \omega_k + 2 \sum_q \eta_{kq} \mu_q^{11} \\
&\quad + 4\Delta^{01} \lambda_k^{03} h^{13} - 4\Delta^{33} \lambda_k^{31} h^{13} + 4\Delta^{10} \lambda_k^{31} h^{30} - 4\Delta^{22} \lambda_k^{03} h^{30} \\
\frac{dh^{13}}{dl} &= \sum_k (-2\kappa_k^{02} \mu_k^{11} \omega_k - 4\Delta^{01} \lambda_k^{03} \mu_k^{11} + 4\Delta^{33} \lambda_k^{31} \mu_k^{11}) \\
\frac{dh^{30}}{dl} &= \sum_k (-2\kappa_k^{21} \mu_k^{11} \omega_k - 4\Delta^{10} \lambda_k^{31} \mu_k^{11} + 4\Delta^{22} \lambda_k^{03} \mu_k^{11}).
\end{aligned} \tag{D1}$$

The first equation for the coefficient  $\Delta^{00}$  constitutes an energy renormalization. It is not necessary to take this equation into account for the numerical integration of the correlation functions. The  $\eta_{kq}$  coefficients are given by ( $k \neq q$ )

$$\begin{aligned}
\eta_{kq} &= \frac{1}{\omega_q^2 - \omega_k^2} \left\{ 2\omega_k \left[ (\lambda_k^{03} \kappa_q^{21} \langle \Sigma_{22} \rangle + \lambda_k^{31} \kappa_q^{02} \langle \Sigma_{33} \rangle - \lambda_k^{03} \kappa_q^{02} \langle \Sigma_{01} \rangle - \lambda_k^{31} \kappa_q^{21} \langle \Sigma_{10} \rangle) \omega_k \right. \right. \\
&\quad + \left. (\lambda_q^{03} \kappa_k^{21} \langle \Sigma_{22} \rangle + \lambda_q^{31} \kappa_k^{02} \langle \Sigma_{33} \rangle - \lambda_q^{03} \kappa_k^{02} \langle \Sigma_{01} \rangle - \lambda_q^{31} \kappa_k^{21} \langle \Sigma_{10} \rangle) \omega_q \right] \\
&\quad - 2\omega_q \left[ (-\kappa_q^{02} \lambda_k^{31} \langle \Sigma_{33} \rangle - \kappa_q^{21} \lambda_k^{03} \langle \Sigma_{22} \rangle + \kappa_q^{02} \lambda_k^{03} \langle \Sigma_{01} \rangle + \kappa_q^{21} \lambda_k^{31} \langle \Sigma_{10} \rangle) \omega_k \right. \\
&\quad + \left. (-\kappa_q^{02} \lambda_k^{31} \langle \Sigma_{33} \rangle - \kappa_q^{21} \lambda_k^{03} \langle \Sigma_{22} \rangle + \kappa_q^{02} \lambda_k^{03} \langle \Sigma_{01} \rangle + \kappa_q^{21} \lambda_k^{31} \langle \Sigma_{10} \rangle) \omega_q \right] \\
&\quad + 4\omega_k (-\Delta^{01} \kappa_k^{02} \kappa_q^{02} \langle \Sigma_{01} \rangle + \Delta^{01} \kappa_k^{02} \kappa_q^{21} \langle \Sigma_{22} \rangle + \Delta^{10} \kappa_k^{21} \kappa_q^{02} \langle \Sigma_{33} \rangle - \Delta^{10} \kappa_k^{21} \kappa_q^{21} \langle \Sigma_{10} \rangle) \\
&\quad + \Delta^{22} \kappa_k^{21} \kappa_q^{02} \langle \Sigma_{01} \rangle - \Delta^{22} \kappa_k^{21} \kappa_q^{21} \langle \Sigma_{22} \rangle - \Delta^{33} \kappa_k^{02} \kappa_q^{02} \langle \Sigma_{33} \rangle + \Delta^{33} \kappa_k^{02} \kappa_q^{21} \langle \Sigma_{10} \rangle) \\
&\quad + 4\omega_k (-\Delta^{01} \kappa_q^{02} \kappa_k^{02} \langle \Sigma_{01} \rangle + \Delta^{01} \kappa_q^{02} \kappa_k^{21} \langle \Sigma_{22} \rangle + \Delta^{10} \kappa_q^{21} \kappa_k^{02} \langle \Sigma_{33} \rangle - \Delta^{10} \kappa_q^{21} \kappa_k^{21} \langle \Sigma_{10} \rangle) \\
&\quad + \Delta^{22} \kappa_q^{21} \kappa_k^{02} \langle \Sigma_{01} \rangle - \Delta^{22} \kappa_q^{21} \kappa_k^{21} \langle \Sigma_{22} \rangle - \Delta^{33} \kappa_q^{02} \kappa_k^{02} \langle \Sigma_{33} \rangle + \Delta^{33} \kappa_q^{02} \kappa_k^{21} \langle \Sigma_{10} \rangle) \\
&\quad + 4\omega_q (-\Delta^{01} \lambda_k^{03} \lambda_q^{03} \langle \Sigma_{01} \rangle + \Delta^{01} \lambda_k^{03} \lambda_q^{31} \langle \Sigma_{33} \rangle + \Delta^{10} \lambda_k^{31} \lambda_q^{03} \langle \Sigma_{22} \rangle - \Delta^{10} \lambda_k^{31} \lambda_q^{31} \langle \Sigma_{10} \rangle) \\
&\quad - \Delta^{22} \lambda_k^{03} \lambda_q^{03} \langle \Sigma_{22} \rangle + \Delta^{22} \lambda_k^{03} \lambda_q^{31} \langle \Sigma_{10} \rangle + \Delta^{33} \lambda_k^{31} \lambda_q^{03} \langle \Sigma_{01} \rangle - \Delta^{33} \lambda_k^{31} \lambda_q^{31} \langle \Sigma_{33} \rangle) \\
&\quad + 4\omega_q (-\Delta^{01} \lambda_q^{03} \lambda_k^{03} \langle \Sigma_{01} \rangle + \Delta^{01} \lambda_q^{03} \lambda_k^{31} \langle \Sigma_{33} \rangle + \Delta^{10} \lambda_q^{31} \lambda_k^{03} \langle \Sigma_{22} \rangle - \Delta^{10} \lambda_q^{31} \lambda_k^{31} \langle \Sigma_{10} \rangle) \\
&\quad - \Delta^{22} \lambda_q^{03} \lambda_k^{03} \langle \Sigma_{22} \rangle + \Delta^{22} \lambda_q^{03} \lambda_k^{31} \langle \Sigma_{10} \rangle + \Delta^{33} \lambda_q^{31} \lambda_k^{03} \langle \Sigma_{01} \rangle - \Delta^{33} \lambda_q^{31} \lambda_k^{31} \langle \Sigma_{33} \rangle) \left. \right\}.
\end{aligned} \tag{D2}$$

For  $\Delta > 0$ , and  $\Delta_S > 0$  the only non-vanishing expectation values  $\langle \Sigma_{\alpha\beta} \rangle$  are

$$\begin{aligned}
\langle \Sigma_{00} \rangle &= \langle \Sigma_{11} \rangle = 1 \\
\langle \Sigma_{01} \rangle &= \langle \Sigma_{10} \rangle = -\frac{\Delta + \Delta_S}{\sqrt{g^2 + (\Delta + \Delta_S)^2}} \\
\langle \Sigma_{22} \rangle &= -\langle \Sigma_{33} \rangle = \frac{g}{\sqrt{g^2 + (\Delta + \Delta_S)^2}}.
\end{aligned} \tag{D3}$$

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